# Filtrations, adapted processes, stopping times

**Def** $(\Omega, \mathcal{F})$ - measurable space. ($\mathcal{F}$ - $\sigma$-algebra, $\subset 2^{\Omega}$)

Filtration is an increasing family $(\mathcal{F}_t)_{t\in I}$, ($\mathcal{F}_t \subset \mathcal{F}_s$, $t \le s$) of sub-$\sigma$-algebras $\mathcal{F}_t \subset \mathcal{F}$. A measurable space with filtration is called filtered space. $I$ can be $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}_+$ or $\mathbb{R}$

**Example** Dyadic filtration.

$\mathcal{B}_n = \sigma\left(\left\{ \left[\frac{k}{2^n}, \frac{\ell}{2^n}\right), \ k<\ell \right\}\right)$, $\mathcal{B}$ - Borel $\sigma$-algebra.

$\sigma(\cup \mathcal{B}_n) = \mathcal{B}$.

**Def** A process $(X_t)_{t\in I}$ is adapted to filtration $(\mathcal{F}_t)$ if $\forall t \in I$, $X_t$ is $\mathcal{F}_t$ measurable.

**Example.** Adapted to dyadic filtration $\iff$ $X_n$ is constant on any $\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)$.

**Def** Given $(X_t)_{t\in I}$, there is the smallest filtration $(\mathcal{F}_t^0)$ such that $\forall t$ $X_t$ is $\mathcal{F}_t^0$-measurable.

Natural filtration. $\mathcal{F}_t^0 = \sigma\{ X_s < a, \ s \le t, \ a \in \mathbb{R} \}$.

**Def** Left, right filtrations

$$\mathcal{F}_t^- := \sigma\Big(\bigcup_{s<t} \mathcal{F}_s\Big), \quad \mathcal{F}_t^+ := \bigcap_{s>t} \mathcal{F}_s, \quad \mathcal{F}_\infty := \sigma\Big(\bigcup_{s\in I} \mathcal{F}_s\Big).$$

**Def** A stopping time with respect to filtration $(\mathcal{F}_t)_{t\in I}$ is a function $T: \Omega \to I \cup \infty$, such that $\forall t \in I$ $\{\omega : T(\omega) \le t\} \in \mathcal{F}_t$

Heuristically: At time $t$ you know whether to stop or continue.

## Examples.

1) $I = \mathbb{N}$. $\quad T = \inf\{n : \underbrace{X_n}_{\text{metric space } E\text{-valued}} \in A\}$ (with $\inf \emptyset = \infty$).
$A \subset E$ $\quad$ hitting time of $A$.
Stopping time in natural filtration.

2) (Continuous hitting time)
$(X_t)_{t \in \mathbb{R}_+}$ (or $t \in \mathbb{R}$) – left-continuous a.s, valued in metric space $E$,
$A \subset E$. $T = \inf\{t : X_t \in A\}$ – stopping time

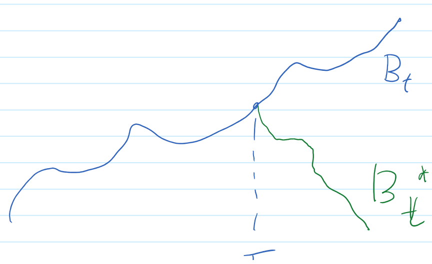
Proof.
$$\{\omega : T(\omega) \le t\} = \{X_t \in A\} \cup \Big(\bigcap_r \bigcup_n \{\underbrace{q \in \mathbb{Q}}_{\text{countable}} : q < t - \tfrac{1}{r} \Rightarrow \operatorname{dist}(X_q, A) < \tfrac{1}{n}\}\Big) \in \mathcal{F}_t$$

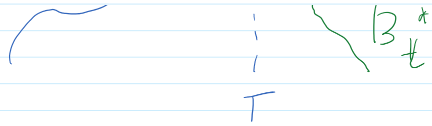
## Stopping time and Brownian Motion.

**Thm.** Let $(B_t)$ be adapted to a filtration $(\mathcal{F}_t)$
(does not need to be the natural filtration!)
$T$ – stopping time with respect to $(\mathcal{F}_t)$, $T < \infty$ a.s.
$B_t^{(T)} := (B(t+T) - B_T)$. Then $B_t^{(T)}$ is a Brownian Motion,
independent of $(B_s)_{s \le T}$

## An application: reflexion principle for BM.

**Thm** Let $T$ be a stopping time, $T < \infty$ a.s.
$$\underbrace{B_t^*}_{\text{Brownian motion.}} := \begin{cases} B_t, & t \le T \\ 2B_T - B_t, & t > T. \end{cases}$$






<u>Proof.</u> Continuity is obvious.

By strong Markov property, both $B_t^1 := B(t+T) - B(T)$ and $B_t^2 := B(T) - B(t+T)$ are BM independent of $(B_s)_{s \le T}$.

Then

$$B_t = \begin{cases} B_t, & t \le T \\ B_T + B^1_{t-T}, & t \ge T \end{cases} \qquad B_t^* = \begin{cases} B_t, & t \le T \\ B_T + B^2_{t-T}, & t \ge T \end{cases}$$

have the same correlations!

<u>Corollary</u> (maximum of Brownian Motion).

Let $S_t = \sup\{s \le t : B_s\}$.

Then for $a \ge 0$, $b \le a$ we have

1) $P(S_t \ge a, B_t \le b) = P(B_t \ge 2a - b)$.

2) $P(S_t \ge a) = P(|B_t| \ge a) = 2P(B_t \ge a)$.

<u>Proof.</u> $T_a := \inf\{t : B_t = a\}$ – stopping time.

Then $P(S_t \ge a, B_t \le b) = P(T_a \le t, B_t \le b) =$

$$P(T_a \le t, B^{(T_a)}_{t-T_a} \le b - a) =$$

$$B^{(T_a)}_{t-T_a} = B_t - B_{T_a} = B_t - a \ (\text{if } t \ge T_a)$$

$$= P(T_a \le t, B_t^* \ge 2a - b) = P(B_t^* \ge 2a - b) = P(B_t \ge 2a - b).$$

reflected wrt $T_a$

since $B_t^* \ge 2a - b \Rightarrow T_a \le t$.

For 2):

$$P(S_t \ge a) = P(S_t \ge a, B_t \ge a) + P(S_t \ge a, B_t \le a) =$$

reflect wrt $T_a$

$$P(S_t \ge a, B_t \ge a) + P(S_t \ge a, B_t^* \ge a) = 2P(B_t \ge a) = P(|B_t| \ge a)$$

$$\underbrace{P(S_t \geq a,\ B_t \geq a)}_{P(B_t \geq a)} + \underbrace{P(S_t \geq a,\ B_t^* \geq a)}_{P(B_t^* \geq a)} = 2P(B_t \geq a) = P(|B_t| \geq a)$$

$\left(\text{Since } \{S_t^* \geq a\} = \{t \leq T_a\} = \{S_t \geq a\}\right)$